

Springs and Circular Motion

October 27, 2025

Springs

Most of the time, we consider a linear spring that follows Hooke's Law.

Hooke's Law: The retarding force of a spring is proportional to its displacement from equilibrium

$$F = -kx.$$

- Here we use $x = \Delta x$ as displacement from equilibrium (rest length), not position.
- Remember that by Newton's 3rd law, the force of a (massless) spring on both sides must be equal.

The potential energy of the spring is the work done to expand it:

$$U = W = \int F \cdot dx = \frac{1}{2}kx^2.$$

Practice Problems

A spring has an equilibrium length of 2.0 meters and a spring constant of 10 newtons/meter. Alice is pulling on one end of the spring with a force of 3.0 newtons. Bob is pulling on the opposite end of the spring with a force of 3.0 newtons, in the opposite direction. What is the resulting length of the spring?

- (A) 1.7 m
- (B) 2.0 m
- (C) 2.3 m
- (D) 2.6 m
- (E) 8.0 m

A block of mass $m = 3.0$ kg is moving on a horizontal surface towards a massless spring with spring constant $k = 80.0$ N/m. The coefficient of kinetic friction between the block and the surface is $\mu_k = 0.50$. The block has a speed of 2.0 m/s when it first comes in contact with the spring. How far will the spring be compressed?

- (A) 0.19 m
- (B) 0.24 m
- (C) 0.39 m
- (D) 0.40 m
- (E) 0.61 m

Two equal masses m are connected by an elastic string that acts like an ideal spring with spring constant k and unstretched length l . The two masses are hung over a frictionless pulley. What is the total length of the string at equilibrium? (Diagram not necessarily to scale.)



- (A) $2mg/k$
- (B) $l + mg/k$
- (C) $l + mg/2k$
- (D) $l + 2mg/k$
- (E) There is not enough information to decide.

Three point masses m are attached together by identical springs. When placed at rest on a horizontal surface the masses form a triangle with side length l . When the assembly is rotated about its center at angular velocity ω , the masses form a triangle with side length $2l$. What is the spring constant k of the springs?

- (A) $2m\omega^2$
- (B) $\frac{2}{\sqrt{3}}m\omega^2$
- (C) $\frac{2}{3}m\omega^2$
- (D) $\frac{1}{\sqrt{3}}m\omega^2$
- (E) $\frac{1}{3}m\omega^2$

Challenge:

A uniform thin circular rubber band of mass M and spring constant k has an original radius R . Now it is tossed into the air. Assume it remains circular when stabilized in air and rotates at angular speed ω about its center uniformly. Which of the following gives the new radius of the rubber band?

- (A) $(2\pi kR)/(2\pi k - M\omega^2)$
- (B) $(4\pi kR)/(4\pi k - M\omega^2)$
- (C) $(8\pi^2 kR)/(8\pi^2 k - M\omega^2)$
- (D) $(4\pi^2 kR)/(4\pi^2 k - M\omega^2)$
- (E) $(4\pi kR)/(2\pi k - M\omega^2)$

Vertical Springs

A spring under the influence of gravity acts equivalently to normal spring, except its equilibrium length is displaced.

Consider a vertical spring of rest length l_0 with one end attached to the ground and the other end attached to a block of mass m . The force on the block is

$$F = -k(x - l_0) - mg = -k\left(x - \left(l_0 - \frac{mg}{k}\right)\right)$$

Or, equilibrium occurs when

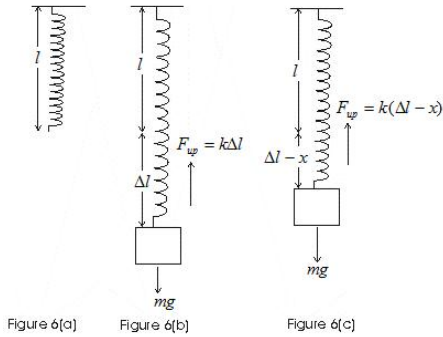
$$k\Delta x = mg \implies \Delta x = \frac{mg}{k}.$$

Thus, it is equivalent to a spring **without gravity** with spring constant k and rest length $l_0 - \frac{mg}{k}$.

What about a vertical spring with one end attached to the ceiling instead? Then, the force on the block is

$$F = -k(x - l_0) + mg = -k\left(x - \left(l_0 + \frac{mg}{k}\right)\right)$$

So it is equivalent to a spring **without gravity** with spring constant k and rest length $l_0 + \frac{mg}{k}$.



- Be careful not to include gravity again when using this model, otherwise you are applying gravity twice.
- Often the situation won't exactly resemble this model, so it's more important to understand how we arrived at this conclusion.

Practice Problems

The following information applies to questions 23 and 24

A man with mass m jumps off of a high bridge with a bungee cord attached to his ankles. The man falls through a maximum distance H at which point the bungee cord brings him to a momentary rest before he bounces back up. The bungee cord is perfectly elastic, obeying Hooke's force law with a spring constant k , and stretches from an original length of L_0 to a final length $L = L_0 + h$. The maximum tension in the Bungee cord is four times the weight of the man.

23. Determine the spring constant k .

- (A) $k = \frac{mg}{h}$
- (B) $k = \frac{2mg}{h}$
- (C) $k = \frac{mg}{H}$
- (D) $k = \frac{4mg}{H}$
- (E) $k = \frac{8mg}{H}$

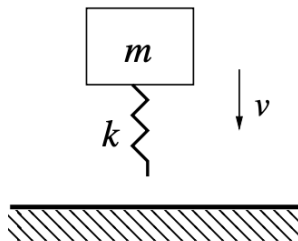
24. Find the maximum extension of the bungee cord h .

- (A) $h = \frac{1}{2}H$
- (B) $h = \frac{1}{4}H$
- (C) $h = \frac{1}{5}H$
- (D) $h = \frac{2}{5}H$
- (E) $h = \frac{1}{8}H$

A 2.0 kg object falls from rest a distance of 5.0 meters onto a 6.0 kg object that is supported by a vertical massless spring with spring constant $k = 72 \text{ N/m}$. The two objects stick together after the collision, which results in the mass/spring system oscillating. What is the maximum magnitude of the displacement of the 6.0 kg object from its original location before it is struck by the falling object?

- (A) 0.27 m
- (B) 1.1 m
- (C) 2.5 m
- (D) 2.8 m
- (E) 3.1 m

A pogo stick is modeled as a massless spring of spring constant k attached to the bottom of a block of mass m . The pogo stick is dropped with the spring pointing downward and hits the ground with speed v . At the moment of the collision, the free end of the spring sticks permanently to the ground.



During the subsequent oscillations, the maximum speed of the block is

- (A) v
- (B) $v + 2mg^2/kv$
- (C) $v + mg^2/kv$
- (D) $\sqrt{v^2 + 2mg^2/k}$
- (E) $\sqrt{v^2 + mg^2/k}$

Combining Springs

Remember that the spring constant is defined as $k = F/x$. When multiple springs are put together, you can determine the spring constant by computing the displacement x when some force F is applied to one end.

Springs in Parallel: For n springs with same equilibrium length, the equivalent spring constant is the sum of the individual spring constants

$$k' = k_1 + k_2 + \cdots + k_n.$$

Proof:

Consider an object connected to two springs in parallel, with spring constants k_1 and k_2 . Then, the net force is

$$F = k_1x + k_2x = (k_1 + k_2)x.$$

Thus, the equivalent spring constant is

$$k' = k_1 + k_2.$$

Springs in Series: For n springs (not necessarily same equilibrium length) connected in series, the reciprocal of the equivalent spring constant is the sum of the reciprocals of the individual spring constants.

$$\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}.$$

Proof:

Consider two springs with spring constants k_1 and k_2 , stretched by a displacement x . The connection point between the two springs must be in equilibrium, since it's not moving, so the force from both springs must be equal. For some applied force F , the displacement (from equilibrium) is

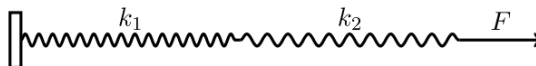
$$x = x_1 + x_2 \implies \frac{F}{k'} = \frac{F}{k_1} + \frac{F}{k_2} \implies \frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2}$$

Practice Problems

A spring stretched to double its unstretched length has a potential energy U_0 . If the spring is cut in half, and each half spring is stretched to double its unstretched length, then the total potential energy stored in the two half springs will be

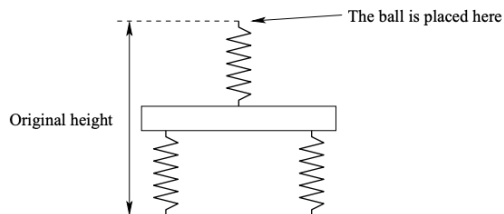
- (A) $4U_0$
- (B) $2U_0$
- (C) U_0
- (D) $U_0/2$
- (E) $U_0/4$

Two springs of spring constants k_1 and k_2 , respectively, are connected in series and stretched, as shown below. What is the ratio of their potential energies, U_1/U_2 ?



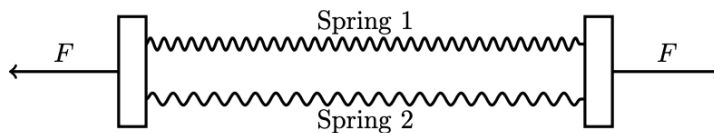
- (A) 1
- (B) k_1/k_2
- (C) k_2/k_1
- (D) $(k_1/k_2)^2$
- (E) $(k_2/k_1)^2$

- . A spring system is set up as follows: a platform with a weight of 10 N is on top of two springs, each with spring constant 75N/m. On top of the platform is a third spring with spring constant 75N/m. If a ball with a weight of 5.0 N is then fastened to the top of the third spring and then slowly lowered, by how much does the height of the spring system change?



- (A) 0.033 m
- (B) 0.067 m
- (C) 0.100 m
- (D) 0.133 m
- (E) 0.600 m

A spring of relaxed length ℓ_1 and spring constant k_1 is placed ‘in parallel’ with a spring of relaxed length ℓ_2 and spring constant k_2 . A force F is applied to each end.

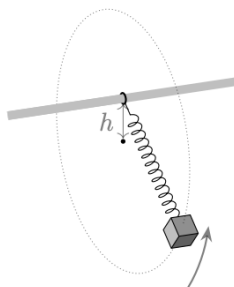


The combination of the springs acts like a single spring with spring constant k and relaxed length ℓ where

- (A) $k = k_1 + k_2$ and $\ell = \ell_1 \ell_2 / (\ell_1 + \ell_2)$
- (B) $k = k_1 + k_2$ and $\ell = (\ell_1 k_1 + \ell_2 k_2) / (k_1 + k_2)$
- (C) $k = k_1 + k_2$ and $\ell = (\ell_1 k_2 + \ell_2 k_1) / (k_1 + k_2)$
- (D) $k = (\ell_1 k_1 + \ell_2 k_2) / (\ell_1 + \ell_2)$ and $\ell = (\ell_1 k_1 + \ell_2 k_2) / (k_1 + k_2)$
- (E) $k = (\ell_2 k_1 + \ell_1 k_2) / (\ell_1 + \ell_2)$ and $\ell = (\ell_1 k_2 + \ell_2 k_1) / (k_1 + k_2)$

Challenge Problems

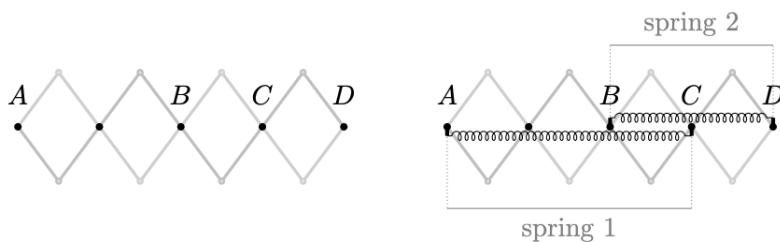
A mass m is connected to one end of a zero-length spring with spring constant k . The other end of the spring is connected to a frictionless bearing mounted around a horizontal pole so that the mass can swing in a vertical circle of radius R around the pole. The setup is shown in the figure below. What is the vertical distance h between the center of the circular orbit and the axis of the pole? Assume that both the diameter of the pole and the rest length of the spring are negligible compared to R .



- (A) $\sqrt{mgR/k}$
- (B) $R\sqrt{(R + mg/k)/(R - mg/k)}$
- (C) $R - mg/k$
- (D) mg/k
- (E) $\sqrt{R^2 - (mg/k)^2}$

Extra Problems

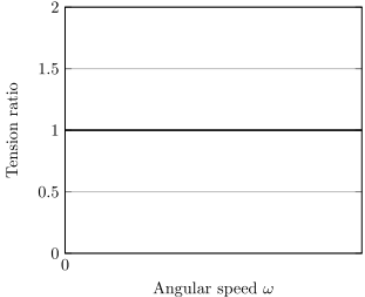
An extendable arm is made from rigid beams free to pivot around the dots shown. Spring 1, with equilibrium length L_1 , is attached between points A and C while spring 2, with equilibrium length L_2 is attached between B and D. The system is allowed to come to equilibrium. In equilibrium, what is the ratio of the tension in spring 1 to the tension in spring 2?



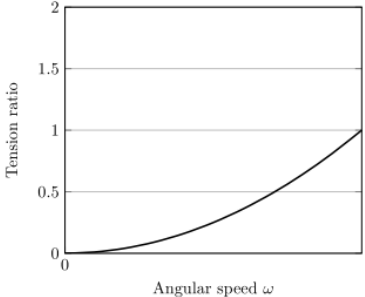
- (A) $-2/3$
- (B) $-3/4$
- (C) -1
- (D) $-3/2$
- (E) There is not enough information to determine the ratio.

A vertical pole has two massless strings, both of length L , attached a distance L apart. The other ends of the strings are attached to a mass M . The mass is rotated around the pole with an angular speed ω . Which of the following graphs best gives the ratio of the tension in the bottom string to the tension in the top string as a function of ω ?

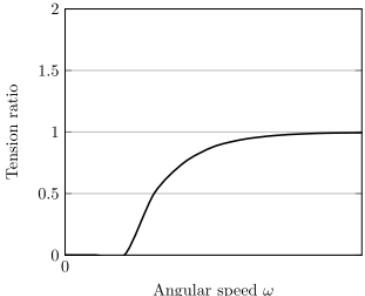
(A)



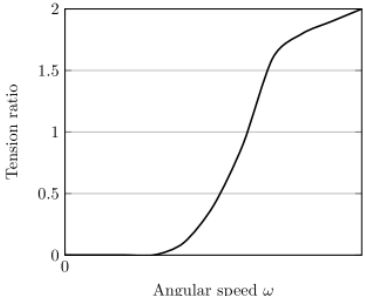
(B)



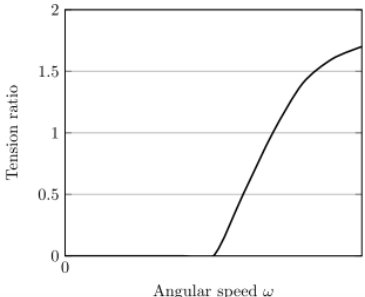
(C)



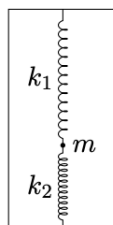
(D)



(E)



A ball of negligible radius and mass m is connected to two ideal springs. Each spring has rest length ℓ_0 . The springs are connected to the ball inside a box of height $2\ell_0$, and the ball is allowed to come to equilibrium, as shown. Under what condition is this equilibrium point stable with respect to small horizontal displacements?



(A) $k_1 > k_2$

(B) $k_2 > k_1$

(C) $k_1 - k_2 > mg/\ell_0$

(D) $k_1 k_2 / (k_1 + k_2) > mg/\ell_0$

(E) $k_1 k_2 / (k_1 - k_2) > mg/\ell_0$

Circular Motion

Definition of Circular Motion: The motion of objects that orbit around a center point.

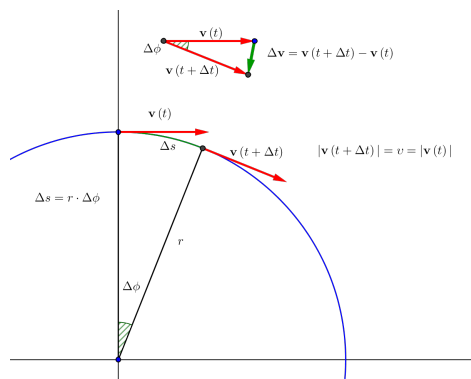
Centripetal Acceleration: Acceleration that causes circular motion.

- Since the velocity of an object in circular orbit constantly changes direction, acceleration must be nonzero. Furthermore, this acceleration must be perpendicular to velocity for the object to have a constant speed (otherwise, the speed will change!).
- The formula for the centripetal acceleration is:

$$a_c = \frac{v^2}{r}$$

where v is the instantaneous velocity and r is the radius of the orbit. Why?

Derivation: Show that the acceleration of a particle moving in a circle is $\frac{v^2}{r}$. Try drawing the position and velocity vectors at two nearby times, then make use of similar triangles.



Centripetal force: Directed toward the center of the circle.

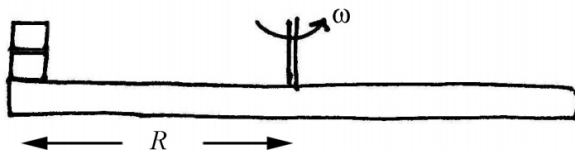
- Not a real force, just a nickname for the total forces pointing to the center:

$$F_c = ma_c = m \frac{v^2}{r}$$

Practice Problems

Two identical coins each of mass m are stacked on top of each other exactly at the rim of a turntable, a distance R from the center. The turntable turns at constant angular speed ω and the coins ride without slipping. Suppose the coefficient of static friction between the turntable and the coin is given by μ_1 and the coefficient of static friction between the coins is given by μ_2 with $\mu_2 < \mu_1$. Let g be the gravitational constant.

- What is the magnitude of the radial force (friction force) exerted by the turntable on the bottom coin?
- As the angular speed increases which coin slips first or do they both slip at the same instant? What is the maximum angular speed $\omega_{\max}$ such that no slipping occurs?



A mass m is attached to a thin rod of length ℓ so that it can freely spin in a vertical circle with period T . The difference in the tensions in the rod when the mass is at the top and the bottom of the circle is

- (A) $6mg^2T^2/\ell$
- (B) $4\pi mg^2T^2/\ell$
- (C) $6mg$
- (D) $\pi^2 m\ell/T^2$
- (E) $4\pi m\ell/T^2$

Tangential acceleration: $a_t = \frac{dv}{dt}$ (change in speed over time), parallel to velocity

- Behaves exactly the same as acceleration in one dimension
- Can apply the 4 kinematics equations if magnitude of a_t is constant

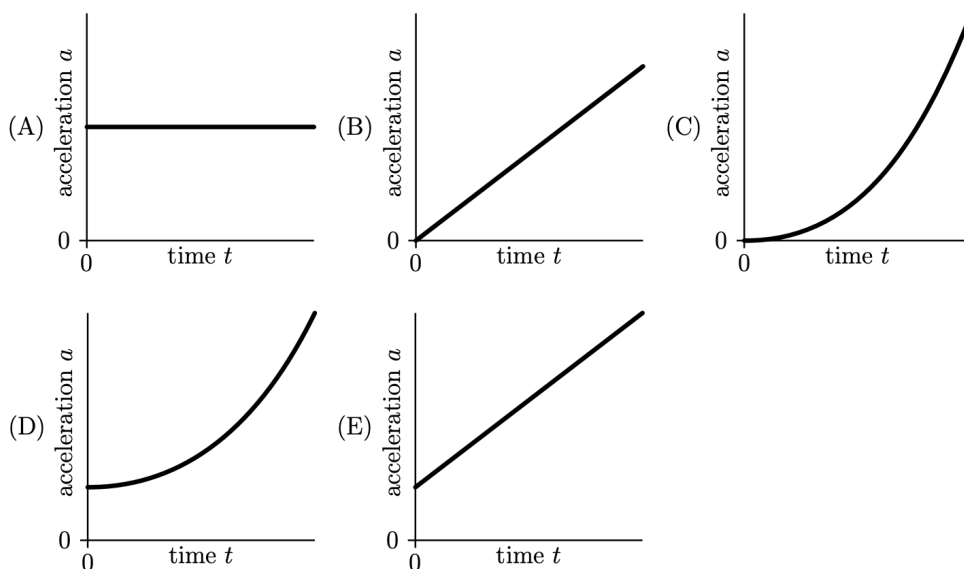
Total acceleration: The vector sum of radial and tangential accelerations:

$$a_{total} = \sqrt{a_c^2 + a_t^2} = \frac{\sum F_{net}}{m}$$

Practice Problems

A mass is initially at rest at the top of a frictionless sphere. It is given a tiny kick and slides off. At what point does it lose contact with the sphere?

4. A small bead slides from rest along a wire that is shaped like a vertical uniform helix (spring). Which graph below shows the magnitude of the acceleration a as a function of time?

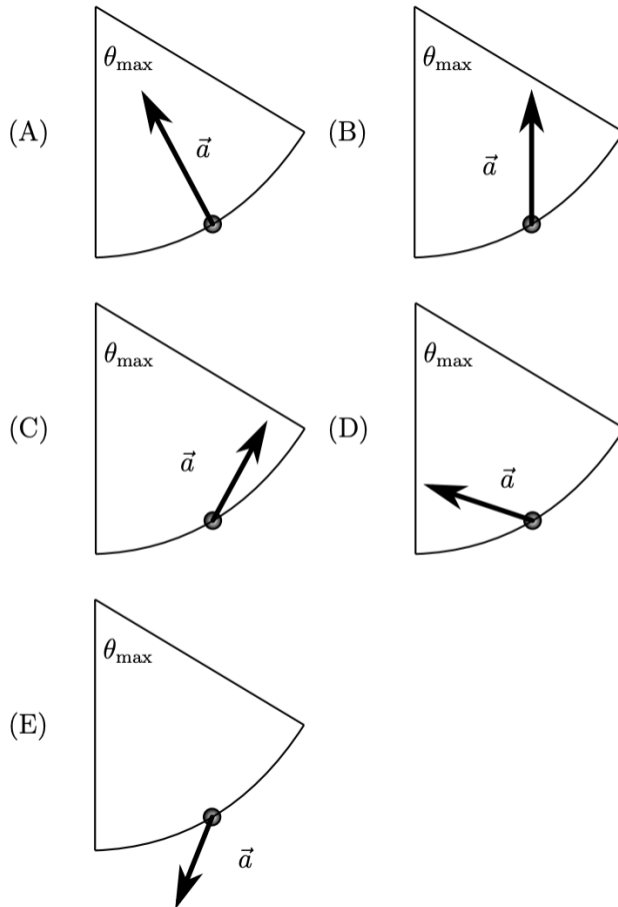


A pendulum consists of a small bob of mass m attached to a fixed point by a string of length L . The pendulum bob swings down from rest from an initial angle $\theta_{\max} < 90$ degrees.

Which of the following statements about the pendulum bob's acceleration is true?

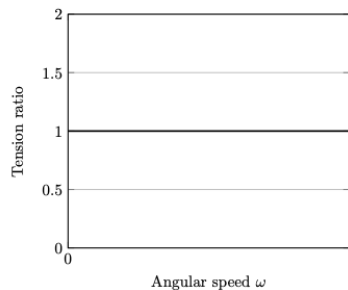
- (A) The magnitude of the acceleration is constant for the motion.
- (B) The magnitude of the acceleration at the lowest point is g , the acceleration of free fall.
- (C) The magnitude of the acceleration is zero at some point of the pendulum's swing.
- (D) The acceleration is always directed toward the center of the circle.
- (E) The acceleration at the bottom of the swing is pointing vertically upward.

Consider the pendulum bob when it is at an angle $\theta = \frac{1}{2}\theta_{\max}$ on the way up (moving toward $\theta_{\max}$). What is the direction of the acceleration vector?

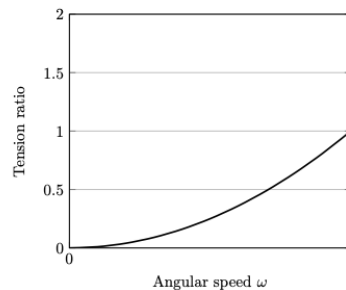


22. A vertical pole has two massless strings, both of length L , attached a distance L apart. The other ends of the strings are attached to a mass M . The mass is rotated around the pole with an angular speed ω . Which of the following graphs best gives the ratio of the tension in the bottom string to the tension in the top string as a function of ω ?

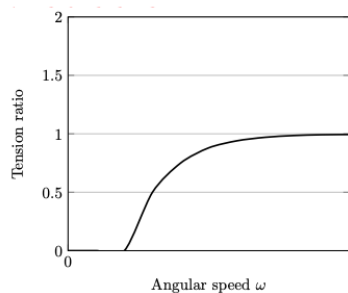
(A)



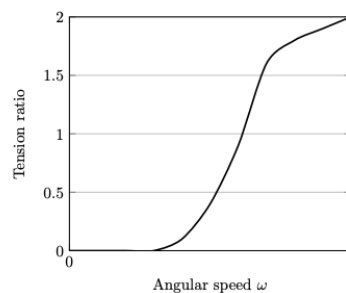
(B)



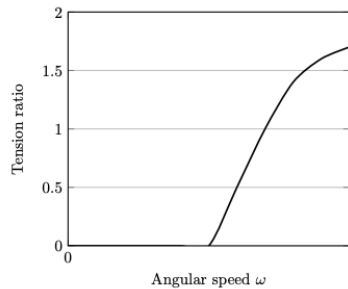
(C)



(D)



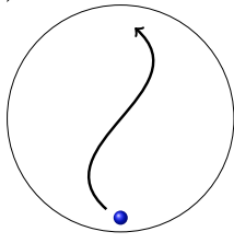
(E)



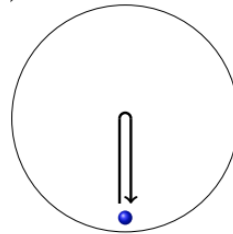
Challenge Problems

12. A child in a circular, rotating space station tosses a ball in such a way so that once the station has rotated through one half rotation, the child catches the ball. From the child's point of view, which plot shows the trajectory of the ball? The child is at the bottom of the space station in the diagrams below, but only the initial location of the ball is shown.

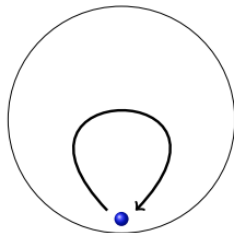
(A)



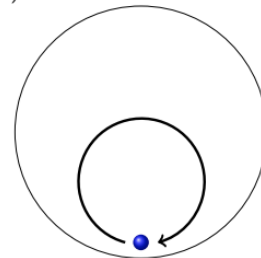
(B)



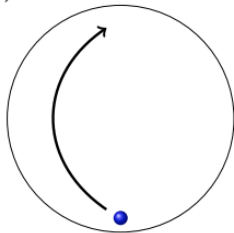
(C)



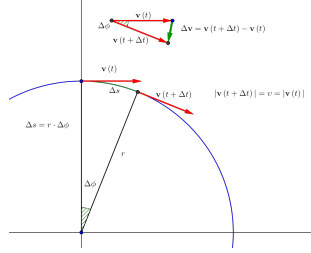
(D)



(E)



Extra Proofs



Derivation of Centripetal Acceleration(calculus proof):

As we can see from the figure above after a time interval Δt , the object has traveled an angle $\Delta\Phi$ in its orbit. Another thing to notice is that the object's final velocity $v(t + \Delta t)$ is equal in magnitude to the initial velocity and makes an angle of Φ with the object's initial velocity. Using the law of cosines to find Δv , we get that Δv is equal to $\sqrt{2v^2 + 2v^2(\cos(\Delta\Phi))}$. We can simplify to get $\Delta v = \sqrt{2}v\sqrt{1 + \cos \Delta\Phi}$. Using the half angle identity, we can simplify $\sqrt{1 + \cos \Delta\Phi}$ to $\sqrt{2} \sin \frac{\Delta\Phi}{2}$. Plugging in that expression into our equation, we get $\Delta v = 2v \sin \frac{\Delta\Phi}{2}$ and using the fact that for small $\Delta\Phi$, $\sin \Delta\Phi = \Delta\Phi$ (Taylor Series Approximation) then we can further simplify our equation to $\Delta v = v\Delta\Phi$. Dividing both sides of the equation by Δt , we get that $\frac{\Delta v}{\Delta t} = a = v \frac{\Delta\Phi}{\Delta t}$. Using the arclength formula $\Delta x = r\Delta\Phi$, we can divide both sides of that equation by Δt to get that $\frac{\Delta x}{\Delta t} = r \frac{\Delta\Phi}{\Delta t}$. We know that $\frac{\Delta x}{\Delta t} = v$ so we then get $\frac{\Delta\Phi}{\Delta t} = \frac{v}{r}$. Plugging that result into $a = v \frac{\Delta\Phi}{\Delta t}$, we finally get $a = \frac{v^2}{r}$.