

Rotation

December 8, 2025

Rotational Kinematics

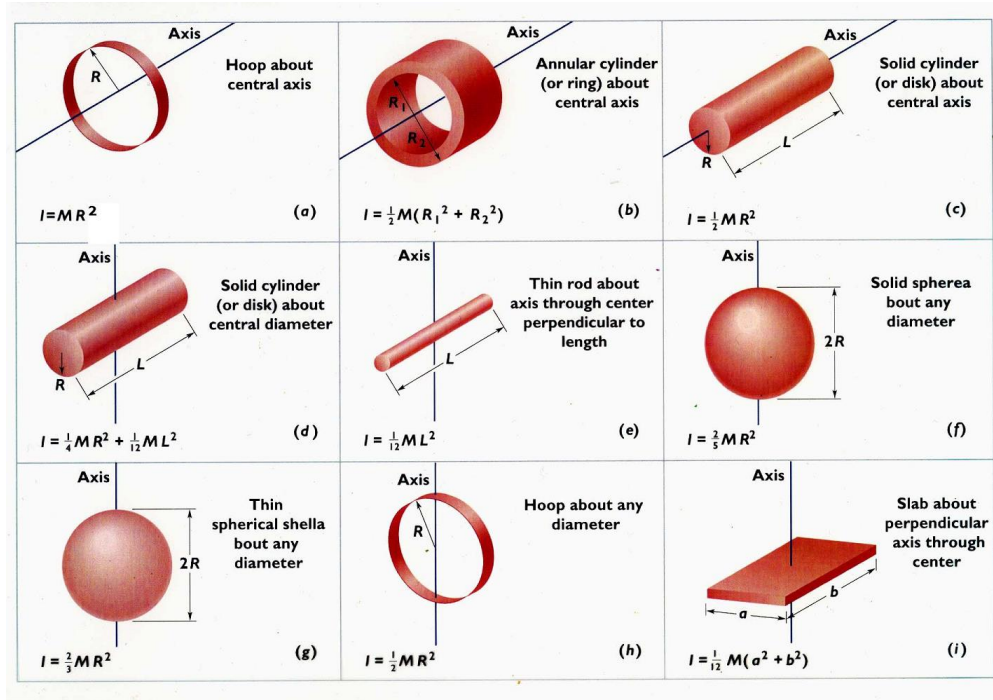
- The sense of how far is traveled is based on the angle in radians rather than the distance in meters
- $x, v, a : \theta, \omega, \alpha$
- Analogous to 1D kinematic equations
 - $\Delta\theta = \omega t + \frac{1}{2}\alpha t^2$
 - $\Delta\omega = \alpha t$
 - $\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta$

Moments of inertia and Fixed Axis Rotation

- Definitions:
 - Rigid Body: every particle on the body has a constant location relative to the other particles in the body, particularly the center of mass
 - Fixed Axis Rotation: rotation where the direction of the axis of rotation is always along the same line, but the axis itself may translate
- Moment of inertia is analogous to mass (resistance to torque/force), but moment of inertia depends on the relative locations of the constituent particles of the rigid body from the axis of rotation

$$I = \int r^2 dm$$

- Common moments of inertia to know for the F=ma Exam



- Parallel Axis Theorem: Can be used to find the moment of inertia of a rigid body of mass M about any axis, given that it is parallel to an axis passing through the center of mass a distance h away from the first.

$$I = I_{cm} + Mh^2$$

- Perpendicular Axis Theorem: For any plane body, the moment of inertia about any of its axes which are perpendicular to the plane is equal to the sum of the moment of inertia about any two perpendicular axes in the plane of the body which intersect the first axis in the plane.

$$I_z = I_x + I_y$$

- Calculating moment of inertia
 - Direct integration - depending on the symmetry, express dm as $dm = \lambda dx$, $dm = \sigma dA$, or $dm = \rho dV$, then integrate.
 - Use parallel axis theorem/perpendicular axis theorem
 - Symmetry - Cutting a shape with a high degree of symmetry into scaled versions of itself, or equivalently cloning the shape into a larger version of itself
 - Slicing a shape into components which are easy to find the moment of inertia of, then summing

Rotational Dynamics

- **Angular momentum $\mathbf{L}$** is closely related to linear momentum $\mathbf{p}$, but one key difference is that angular momentum depends on the axis of rotation/the coordinate system.

$$\mathbf{L} = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum \mathbf{r}_i \times m_i \mathbf{v}_i$$

- Spin and Orbital angular momentum: another perspective of angular momentum

$$\mathbf{L} = I\omega = I_{cm}\omega + \mathbf{R} \times M\mathbf{V}$$

where I is the moment of inertia of the rigid body with respect to the axis of rotation, $\mathbf{R}$ is the position of the center of mass, M is the total mass of the rigid body system, and $\mathbf{V}$ is the center of mass velocity.

In the right-most expression, note that the first term is only considering the rotation about the center of mass (spin) and the second is only considering the movement of the center of mass itself (orbital). This is a useful separation, as we shall see.

- **Torque τ** is also related to force $\mathbf{F}$, with the distinction being the same as angular momentum vs. linear momentum.

$$\begin{aligned}\tau &= \mathbf{r} \times \mathbf{F} \\ \tau &= \frac{d\mathbf{L}}{dt} = I\alpha\end{aligned}$$

Therefore the magnitude of the torque can be written as

$$|\mathbf{r} \times \mathbf{F}| = rF\sin(\theta)$$

Note that there is an analogous expression for torque similar to the spin and orbital angular momentum expression, but since rotational motion about the center of mass is independent of the translational motion, we usually don't have to worry about the other term (and it complicates things). In this simplified consideration, the moment of inertia I would be about the center of mass and the torque would be the net (external) torque about an axis passing through the center of mass.

- Note that $\tau = I\alpha$ strongly parallels $\mathbf{F} = M\mathbf{a}$. In fact, the torque equation is often called Newton's Second Law for Rotation. These two equations are often used concurrently to completely describe the motion of a rigid body.

- **Conservation of Angular momentum**

Note that when the net external torque $\tau = 0$ (isolated system), angular momentum is conserved. This is another powerful conservation law alongside conservation of linear momentum and energy.

- **Problems involving translational and rotational motion**

When solving these problems, it is always possible to break the motion down into independent translation of the center of mass (linear motion) and rotation about the center of mass equations.

One can use, possibly along with conservation of energy, the force (force and torque) perspective or momentum (momentum and angular momentum) perspective - both are valid but one may be easier than the other. Examples for thought:

- Find angular acceleration - use torque
- Find the final angular velocity - use angular momentum conservation (torque works too but then you have to take time into account - so in general momentum/angular momentum evades the consideration of time)

- **Work-Energy Theorem and Kinetic Energy of a Rigid Body**

- The Work-Energy Theorem still applies for rotation - in fact the theorem holds for translation motion and rotational motion separately (or together - up to you):

$$W = \Delta K$$

where W is the total work (done by forces and/or torques) and ΔK is the change in kinetic energy.

- Kinetic energy of a rotating and translating body can be written as

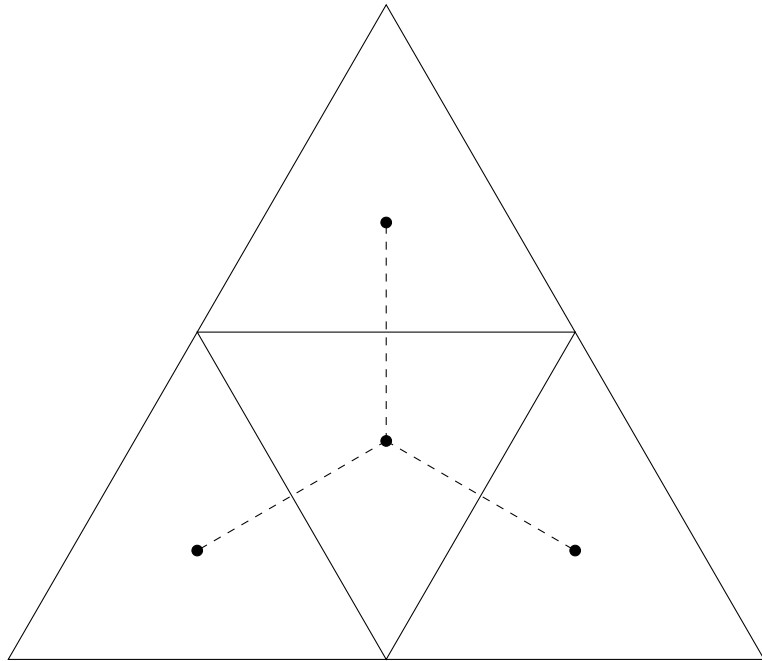
$$K = \frac{1}{2}MV^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}I\omega^2$$

- **Rolling without slipping**

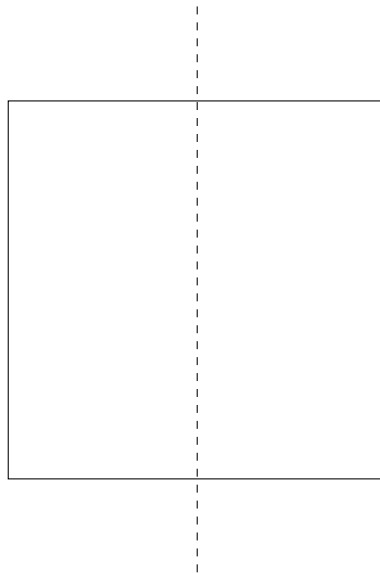
- Rolling without slipping of a rigid body is a special case of a rolling object where the relative velocity between the point touching the ground and the ground is 0. In other words, the velocity of the bottom-most point is $v=0$! Because there is no relative motion at the contact point, if there is friction it must be static.
- On the other hand, rolling with slipping implies that there is kinetic friction.
- This kind of problem is a great example of using the force/torque perspective vs momentum/angular momentum perspective. Note that regardless of whether the rigid body is rolling with or without slipping, angular momentum about an axis on the ground must be conserved, because any friction with the ground would act parallel to the line joining the axis and the contact point.

Problems

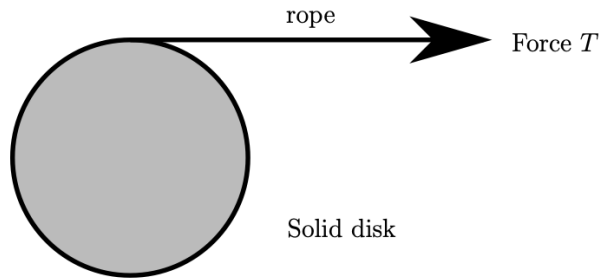
Consider finding the moment of inertia of a triangle with mass M and side length l . One method is to integrate of many small strips of the triangle that are parallel to a base. A simpler way is to divide the triangle into four smaller triangles (or clone the triangle).....Find the moment of inertia this way.



Find the moment of inertia of a square sheet with mass M and side length l about an axis in the plane of the sheet, parallel to its sides, and passing through the center.



A disk of moment of inertia I , mass M , and radius R has a cord wrapped around it tightly as shown in the diagram. The disk is free to slide on its side as shown in the top down view. A constant force of T is applied to the end of the cord and accelerates the disk along a frictionless surface.



After the disk has accelerated some distance, determine the ratio of the translational KE to total KE of the disk,

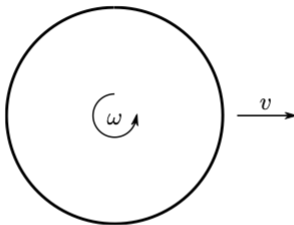
$$KE_{\text{translational}}/KE_{\text{total}} =$$

- (A) $\frac{I}{MR^2}$
- (B) $\frac{MR^2}{I}$
- (C) $\frac{I}{3MR^2}$
- (D) $\frac{I}{MR^2+I}$
- (E) $\frac{MR^2}{MR^2+I}$

A crew of scientists has built a new space station. The space station is shaped like a wheel of radius R , with essentially all its mass M at the rim. When the crew arrives, the station will be set rotating at a rate that causes an object at the rim to have radial acceleration g , thereby simulating Earth's surface gravity. This is accomplished by two small rockets, each with thrust T newtons, mounted on the station's rim. How long a time t does one need to fire the rockets to achieve the desired condition?

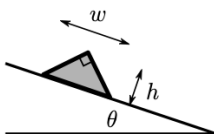
- (A) $t = \sqrt{gR^3} M/(2T)$
- (B) $t = \sqrt{gR} M/(2T)$
- (C) $t = \sqrt{gR} M/T$
- (D) $t = \sqrt{gR/\pi} M/T$
- (E) $t = \sqrt{gR} M/(\pi T)$

As shown in the figure, a ping-pong ball with mass m with initial horizontal velocity v and angular velocity ω comes into contact with the ground. Friction is not negligible, so both the velocity and angular velocity of the ping-pong ball changes. What is the critical velocity v_c such that the ping-pong will stop and remain stopped? Treat the ping-pong ball as a hollow sphere.



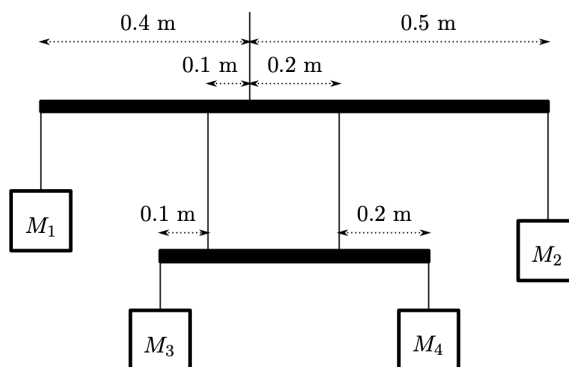
- (A) $v = \frac{2}{3}R\omega$
- (B) $v = \frac{2}{5}R\omega$
- (C) $v = R\omega$
- (D) $v = \frac{3}{5}R\omega$
- (E) $v = \frac{5}{3}R\omega$

A uniform solid right prism whose cross section is an isosceles right triangle with height h and width $w = 2h$ is placed on an incline that has a variable angle with the horizontal θ . What is the minimum coefficient of static friction so that the prism topples before it begins sliding as θ is slowly increased from zero?



- (A) 0.71
- (B) 1.41
- (C) 1.50
- (D) 1.73
- (E) 3.00

In the mobile below, the two cross beams and the seven supporting strings are all massless. The hanging objects are $M_1 = 400$ g, $M_2 = 200$ g, and $M_4 = 500$ g. What is the value of M_3 for the system to be in static equilibrium?



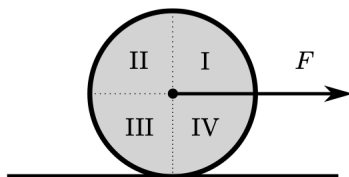
- (A) 300 g
- (B) 400 g
- (C) 500 g
- (D) 600 g
- (E) 700 g

The following information applies to questions 14 and 15.

An object starting from rest can roll without slipping down an incline.

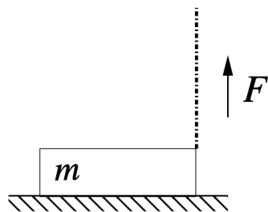
14. Which of the following four objects, each with mass M and radius R , would have the largest acceleration down the incline?
 - (A) A uniform solid sphere
 - (B) A uniform solid disk
 - (C) A hollow spherical shell
 - (D) A hoop
 - (E) All objects would have the same acceleration.
15. Which of the following four objects, each a uniform solid sphere released from rest, would have the largest speed after the center of mass has moved through a vertical distance h ?
 - (A) A sphere of mass M and radius R .
 - (B) A sphere of mass $2M$ and radius $\frac{1}{2}R$.
 - (C) A sphere of mass $M/2$ and radius $2R$.
 - (D) A sphere of mass $3M$ and radius $3R$.
 - (E) All objects would have the same speed.

A uniform disk is being pulled by a force F through a string attached to its center of mass. Assume that the disk is rolling smoothly without slipping. At a certain instant of time, in which region of the disk (if any) is there a point moving with zero total acceleration?



- (A) Region I
- (B) Region II
- (C) Region III
- (D) Region IV
- (E) All points on the disk have non-zero acceleration.

A uniform stick of mass m is originally on a horizontal surface. One end is attached to a vertical rope, which pulls up with a constant tension force F so that the center of the mass of the stick moves upward with acceleration $a < g$. The normal force N of the ground on the other end of the stick shortly after the right end of the stick leaves the surface satisfies



- (A) $N = mg$
- (B) $mg > N > mg/2$
- (C) $N = mg/2$
- (D) $mg/2 > N > 0$
- (E) $N = 0$

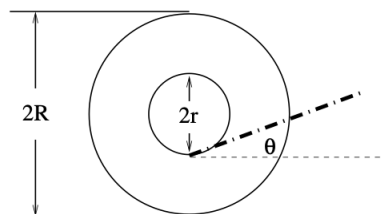
A ping-pong ball (a hollow spherical shell) with mass m is placed on the ground with initial velocity v_0 and zero angular velocity at time $t = 0$. The coefficient of friction between the ping-pong ball and the ground is $\mu_s = \mu_k = \mu$. The time the ping-pong ball begins to roll without slipping is



- (A) $t = (2/5)v_0/\mu g$
- (B) $t = (2/3)v_0/\mu g$
- (C) $t = v_0/\mu g$
- (D) $t = (5/3)v_0/\mu g$
- (E) $t = (3/2)v_0/\mu g$

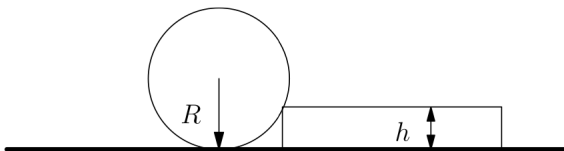
A spool is made of a cylinder with a thin disc attached to either end of the cylinder, as shown. The cylinder has radius $r = 0.75$ cm and the discs each have radius $R = 1.00$ cm. A string is attached to the cylinder and wound around the cylinder a few times. At what angle above the horizontal can the string be pulled so that the spool will slip without rotating?

side-view



- (A) 31.2°
- (B) 41.4°
- (C) 54.0°
- (D) 60.8°
- (E) 81.5°

A cylinder has a radius R and weight G . You try to roll it over a step of height $h < R$. The minimum force needed to roll the cylinder over is:

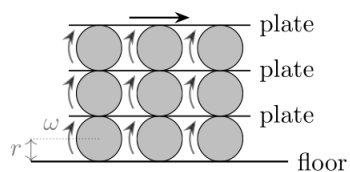


- (A) $\frac{\sqrt{2Rh - h^2}}{R - h} G$
- (B) $\frac{\sqrt{2Rh - h^2}}{2R - h} G$
- (C) $\frac{\sqrt{2Rh - h^2}}{2R} G$
- (D) $\frac{\sqrt{2Rh - h^2}}{R} G$
- (E) $\frac{\sqrt{Rh - h^2}}{2R} G$

An upright rod of length ℓ is launched into the air with vertical velocity v_y . It is given enough angular momentum so that the rod rotates by an angle of 2π before landing. Find the initial horizontal velocity of the bottom of the rod.

- (A) $\frac{2\ell g}{v_y^2}$
- (B) $\frac{\pi \ell g}{2v_y}$
- (C) $\sqrt{\ell g}$
- (D) $\frac{2v_y}{\ell g}$
- (E) $\frac{2v_y^2}{\pi \sqrt{\ell g}}$

A system of cylinders and plates is set up as shown. The cylinders all have radius r , and roll without slipping to the right with angular velocity ω . What is the speed of the top plate?



- (A) ωr
- (B) $2\omega r$
- (C) $3\omega r$
- (D) $4\omega r$
- (E) $6\omega r$