

Problem solving

VCS USAPhO Club 2022-23

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1 Math

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- **Approximations:** A function can be approximated by its Taylor series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2}f''(x_0) + O((x - x_0)^3)$$

Where $O(x^n)$ denotes an error term that grows at most as fast as x^n . Some important examples to know:

$$(1 + x)^n \approx 1 + nx$$

$$\sin x \approx x + x^3/3! + O(x^5) \approx x + O(x^3)$$

$$\cos x \approx 1 - x^2/2 + O(x^4) \approx 1 + O(x^2)$$

$$e^x \approx x + O(x^2)$$

- **Limiting cases:** Think of extreme cases of a situation/formula to determine if it's physically realistic. For example, we know the range of a projectile is 0 if our launch angle is 0° or 90° , so we can remember that the range is proportional to $\sin 2\theta$.
- **Estimation:** Estimate quantities to the nearest order of magnitude see if a value is realistic, e.g. let $g = 10$, the height of a human be 1 meter, a cow be spherical. Computed values should make sense physically. A velocity shouldn't be faster than the speed of light. A temperature shouldn't be below 0 Kelvin.
- **Error propagation:** During experiments, we can't perfectly measure a value, so we associate an error with the measurement: $x \pm \Delta x$. We define absolute uncertainty as Δx and relative uncertainty as $\frac{\Delta x}{x}$

Combining errors: Let us have two measurements $x \pm \Delta x$ and $y \pm \Delta y$.

$$\Delta(ax) = a\Delta x$$

$$\Delta(x + y) = \Delta(x - y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{\Delta(xy)}{xy} = \frac{\Delta(x/y)}{x/y} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

$$\frac{\Delta(x^a)}{x^a} = |a| \frac{\Delta x}{x}$$

$$\Delta f(x, y, z) = \sqrt{\left(\frac{\partial f}{\partial x} \Delta x\right)^2 + \left(\frac{\partial f}{\partial y} \Delta y\right)^2 + \left(\frac{\partial f}{\partial z} \Delta z\right)^2}$$

2 Dimensional Analysis

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- Dimensions should match on both sides of a physical equation.
- Some dimensions include: length (L), mass (M), time (T), charge (Q)
- Example: $d = vt$ $[d] = L$, $[v] = L/T$, $t = T$.
- Can't necessarily tell what is right, but can tell what must be wrong
- Ex. Is the period of a pendulum proportional to $\sqrt{l/g}$ or $\sqrt{g/l}$?

Practice Problems

A mass m moving at speed v collides elastically with a mass M . Derive the expressions for their velocities after the collision using limiting cases.

Alice and Bob are working on a lab report. Alice measures the period of a pendulum to be 1.013 ± 0.008 s, while Bob independently measures the period to be 0.997 ± 0.032 s. Alice and Bob can combine their measurements in several ways.

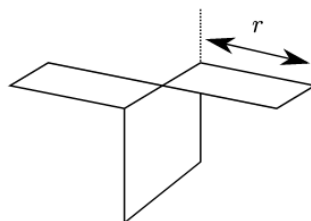
1. Keep Alice's result and ignore Bob's
2. Average Alice's and Bob's results
3. Perform a weighted average of Alice's and Bob's results, with Alice's result weighted 2 times as much as Bob's

How are the uncertainties of these results related?

- a. Method 1 has the lowest uncertainty, and method 2 has the highest
- b. Method 3 has the lowest uncertainty, and method 2 has the highest
- c. Method 2 has the lowest uncertainty, and method 1 has the highest
- d. Method 3 has the lowest uncertainty, and method 1 has the highest
- e. Method 1 has the lowest uncertainty, and method 3 has the highest

2 DIMENSIONAL ANALYSIS

A paper helicopter with rotor radius r and weight W is dropped from a height h in air with a density of ρ .



Assuming that the helicopter quickly reaches terminal velocity, a function for the time of flight T can be found in the form

$$T = kh^{\alpha}r^{\beta}\rho^{\delta}W^{\omega}.$$

where k is an unknown dimensionless constant (actually, 1.164). α , β , δ , and ω are constant exponents to be determined.

12. Determine α .

- (A) $\alpha = -1$
- (B) $\alpha = -1/2$
- (C) $\alpha = 0$
- (D) $\alpha = 1/2$
- (E) $\alpha = 1$

13. Determine β .

- (A) $\beta = 1/3$
- (B) $\beta = 1/2$
- (C) $\beta = 2/3$
- (D) $\beta = 1$
- (E) β can not be uniquely determined without more information.

A ball is thrown at an angle θ up to the top of a cliff of height L , from a point a distance L from the base, as shown in Fig. 1.9. Assuming that one of the following quantities is the initial speed required to make the ball hit right at the edge of the cliff, which one is it? (Don't solve the problem from scratch, just check special cases.)

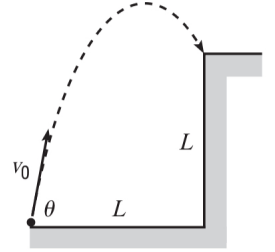


Fig. 1.9

$$\sqrt{\frac{gL}{2(\tan \theta - 1)}}, \quad \frac{1}{\cos \theta} \sqrt{\frac{gL}{2(\tan \theta - 1)}}, \quad \frac{1}{\cos \theta} \sqrt{\frac{gL}{2(\tan \theta + 1)}},$$

$$\sqrt{\frac{gL \tan \theta}{2(\tan \theta + 1)}}.$$

- 1.17. A person throws a ball with a given speed v (at the optimal angle for the following task) toward a wall of height h . Which of the following quantities is the maximum distance the person can stand from the wall and still be able to throw the ball over the wall?

(a) $\frac{gh^2}{v^2}$ (b) $\frac{v^2}{g}$ (c) $\frac{v^4}{g^2h}$ (d) $\sqrt{\frac{v^2h}{g}}$ (e) $\frac{v^2}{g} \sqrt{1 - \frac{2gh}{v^2}}$ (f) $\frac{v^2/g}{1 + 2gh/v^2}$.

17. A flywheel can rotate in order to store kinetic energy. The flywheel is a uniform disk made of a material with a density ρ and tensile strength σ (measured in Pascals), a radius r , and a thickness h . The flywheel is rotating at the maximum possible angular velocity so that it does not break. Which of the following expression correctly gives the maximum kinetic energy per kilogram that can be stored in the flywheel? Assume that α is a dimensionless constant.

- (A) $\alpha \sqrt{\rho \sigma / r}$
 (B) $\alpha h \sqrt{\rho \sigma / r}$
 (C) $\alpha \sqrt{(h/r)(\sigma/\rho)^2}$
 (D) $\alpha (h/r)(\sigma/\rho)$
 (E) $\alpha \sigma / \rho$