

# Oscillations

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## Review

In most oscillation problems, we try to find the period of some system in simple harmonic motion.

- **Simple harmonic motion:** periodic motion where restoring force is proportional to displacement from equilibrium; i.e.  $F \propto a \propto x$ , so

$$a = -bx,$$

for some proportionality constant  $b$ . This is typically written as

$$\ddot{x} + bx = 0,$$

where  $\ddot{x} = a$  is the second derivative of  $x$  with respect to time. The solution to this differential equation is a sine or cosine function.

Let  $x = A \sin(\omega t)$ , for some constant angular frequency  $\omega$ . Then,

$$\ddot{x} = -A\omega^2 \sin(\omega t) = -b(A \sin(\omega t))$$

Thus,

$$b = \omega^2.$$

- The period and frequency of a system in simple harmonic motion is

$$T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi}$$

- Since  $x$  is sinusoidal, its derivatives, velocity  $v$  and acceleration  $a$  must also be sinusoidal. The respective equations are

$$x = A \sin(\omega t)$$

$$v = A\omega \cos(\omega t)$$

$$a = -A\omega^2 \sin(\omega t).$$

If the oscillation starts from a different position, there may be a phase shift constant  $\phi$ , giving the solution

$$\begin{aligned}x &= A \sin(\omega t + \phi). \\v &= A\omega \cos(\omega t + \phi) \\a &= -A\omega^2 \sin(\omega t + \phi).\end{aligned}$$

- A constant acceleration (such as gravity acting on a vertical spring) does not affect the oscillation period. Thus, more generally, the simple harmonic motion equation is

$$\ddot{x} = -\omega^2 x + c \iff \ddot{x} + \omega^2 x = c,$$

which has the solution

$$x = \frac{c}{\omega^2} + A \sin(\omega t + \phi).$$

(Why is this true?) Thus the period is the same:  $T = \frac{2\pi}{\omega}$ . The position  $x$  is shifted by a constant of  $\frac{c}{\omega^2}$ .

## Example derivations

- **Spring:** For a linear spring following Hooke's Law, the retarding force is  $F = -kx$ . (This  $k$  is different from the constant above). Then,

$$ma = -kx$$

$$a = -\frac{k}{m}x.$$

We know the solution to this is

$$x = A \sin(\omega t + \phi),$$

where  $\omega^2 = \frac{k}{m}$ , so  $\omega = \sqrt{\frac{k}{m}}$ . Thus, its period is

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}.$$

If the spring starts from equilibrium,  $\phi = 0$  and its motion is a simple sine wave.

- **Simple Pendulum** (massless string): This time we consider the angular displacement. Tension is radial, so it provides no torque. Only force with torque is gravity.

$$\tau = L \times F = -mgL \sin(\theta),$$

where  $\theta$  is the angular displacement from the vertical.

$$I\alpha = -mgL \sin(\theta).$$

$$mL^2\alpha = -mgL\sin(\theta)$$

$$\alpha = -\frac{g}{L}\sin(\theta).$$

For small  $\theta$ ,  $\sin \theta \approx \theta$ , so

$$\ddot{\theta} = -\frac{g}{L}\theta$$

$$\omega^2 = \frac{g}{L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}.$$

- **Physical Pendulum:** For a simple pendulum, the center of mass (where the torque from gravity effectively acts) is located distance  $L$  from the pivot, so the moment of inertia was  $mL^2$ . For a physical pendulum such as a hanging rigid object, the moment of inertia  $I$  about the pivot axis depends on the radial distribution of mass. Similarly,

$$I\alpha = -mgL\sin(\theta) \approx -mgl\theta$$

$$I\ddot{\theta} + mgL\theta = 0.$$

$$\omega^2 = \frac{mgL}{I}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}}$$

### Practice Problems

A light, uniform, ideal spring is fixed at one end. If a mass is attached to the other end, the system oscillates with angular frequency  $\omega$ . Now suppose the spring is fixed at the other end, then cut in half. The mass is attached between the two half springs.



The new angular frequency of oscillations is

- (A)  $\omega/2$
- (B)  $\omega$
- (C)  $\sqrt{2}\omega$
- (D)  $2\omega$
- (E)  $4\omega$

An object of mass  $M$  is hung on a vertical spring of spring constant  $k$  and is set into vertical oscillations. The period of this oscillation is  $T_0$ . The spring is then cut in half and the same mass is attached and the system is set up to oscillate on a frictionless inclined plane making an angle  $\theta$  to the horizontal. Determine the period of the oscillations on the inclined plane in terms of  $T_0$ .

- (A)  $T_0$
- (B)  $T_0/2$
- (C)  $2T_0 \sin \theta$
- (D)  $T_0/\sqrt{2}$
- (E)  $T_0 \sin \theta/\sqrt{2}$

A pendulum of length  $L$  oscillates inside a box. A person picks up the box and gently shakes it vertically with frequency  $\omega$  and a fixed amplitude for a fixed time. To maximize the final amplitude of the pendulum,  $\omega$  should satisfy

- (A)  $\omega = \sqrt{4g/L}$
- (B)  $\omega = \sqrt{2g/L}$
- (C)  $\omega = \sqrt{g/L}$
- (D)  $\omega = \sqrt{g/4L}$
- (E) there will be no significant effect on the pendulum amplitude for any value of  $\omega$

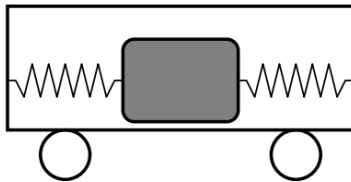
Two identical objects of mass  $m$  are placed at either end of a spring of spring constant  $k$  and the whole system is placed on a horizontal frictionless surface. At what angular frequency  $\omega$  does the system oscillate?

- (A)  $\sqrt{k/m}$
- (B)  $\sqrt{2k/m}$
- (C)  $\sqrt{k/2m}$
- (D)  $2\sqrt{k/m}$
- (E)  $\sqrt{k/m}/2$

A small rock is tied to a massless string of length 5 m. The density of the rock is twice the density of the water. The rock is lowered into the water, while the other end of the string is attached to a pivot. Neglect any resistive forces from the water. The rock oscillates like a pendulum with angular frequency of which of the following?

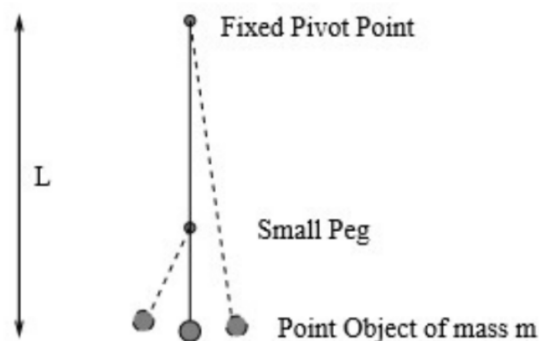
- (A) 1 rad/s
- (B) 0.7 rad/s
- (C) 0.5 rad/s
- (D) 1.4 rad/s
- (E) 2 rad/s

Inside a cart that is accelerating horizontally at acceleration  $\vec{a}$ , there is a block of mass  $M$  connected to two light springs of force constants  $k_1$  and  $k_2$ . The block can move without friction horizontally. Find the vibration frequency of the block.



- (A)  $\frac{1}{2\pi} \sqrt{\frac{k_1 + k_2}{M}} + a$
- (B)  $\frac{1}{2\pi} \sqrt{\frac{k_1 k_2}{(k_1 + k_2)M}}$
- (C)  $\frac{1}{2\pi} \sqrt{\frac{k_1 k_2}{(k_1 + k_2)M}} + a$
- (D)  $\frac{1}{2\pi} \sqrt{\frac{|k_1 - k_2|}{M}}$
- (E)  $\frac{1}{2\pi} \sqrt{\frac{k_1 + k_2}{M}}$

A simple pendulum of length  $L$  is constructed from a point object of mass  $m$  suspended by a massless string attached to a fixed pivot point. A small peg is placed a distance  $2L/3$  directly below the fixed pivot point so that the pendulum would swing as shown in the figure below. The mass is displaced 5 degrees from the vertical and released. How long does it take to return to its starting position?



- (A)  $\pi\sqrt{\frac{L}{g}}\left(1 + \sqrt{\frac{2}{3}}\right)$
- (B)  $\pi\sqrt{\frac{L}{g}}\left(2 + \frac{2}{\sqrt{3}}\right)$
- (C)  $\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{3}\right)$
- (D)  $\pi\sqrt{\frac{L}{g}}\left(1 + \sqrt{3}\right)$
- (E)  $\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{\sqrt{3}}\right)$

Two identical carts A and B each with mass  $m$  are connected via a spring with spring constant  $k$ . Two additional springs, identical to the first, connect the carts to two fixed points. The carts are free to oscillate under the effect of the springs in one dimensional frictionless motion.



Under suitable initial conditions, the two carts will oscillate in phase according to

$$x_A(t) = x_0 \sin \omega_1 t = x_B(t)$$

where  $x_A$  and  $x_B$  are the locations of carts A and B relative to their respective equilibrium positions.

Under other suitable initial conditions, the two carts will oscillate exactly out of phase according to

$$x_A(t) = x_0 \sin \omega_2 t = -x_B(t)$$

Determine the ratio  $\omega_2/\omega_1$

- (A)  $\sqrt{3}$
- (B) 2
- (C)  $2\sqrt{2}$
- (D) 3
- (E) 5

A simple pendulum experiment is constructed from a point mass  $m$  attached to a pivot by a massless rod of length  $L$  in a constant gravitational field. The rod is released from an angle  $\theta_0 < \pi/2$  at rest and the period of motion is found to be  $T_0$ . Ignore air resistance and friction.

The experiment is repeated with a new pendulum with a rod of length  $4L$ , using the same angle  $\theta_0$ , and the period of motion is found to be  $T$ . Which of the following statements is correct?

- (A)  $T = 2T_0$  regardless of the value of  $\theta_0$ .
- (B)  $T > 2T_0$  with  $T \approx 2T_0$  if  $\theta_0 \ll 1$ .
- (C)  $T < 2T_0$  with  $T \approx 2T_0$  if  $\theta_0 \ll 1$ .
- (D)  $T > 2T_0$  for some values of  $\theta_0$  and  $T < 2T_0$  for other values of  $\theta_0$ .
- (E)  $T_0$  and  $T$  are undefined because the motion is not periodic unless  $\theta_0 \ll 1$ .

A U-tube manometer consists of a uniform diameter cylindrical tube that is bent into a U shape. It is originally filled with water that has a density  $\rho_w$ . The *total* length of the column of water is  $L$ . Ignore surface tension and viscosity.

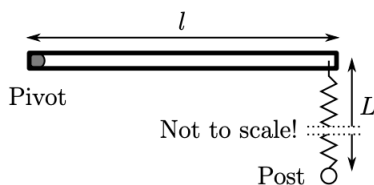
The water is displaced slightly so that one side moves up a distance  $x$  and the other side lowers a distance  $x$ . Find the frequency of oscillation.

- (A)  $\frac{1}{2\pi}\sqrt{2g/L}$
- (B)  $2\pi\sqrt{g/L}$
- (C)  $\frac{1}{2\pi}\sqrt{2L/g}$
- (D)  $\frac{1}{2\pi}\sqrt{g/\rho_w}$
- (E)  $2\pi\sqrt{\rho_w g L}$

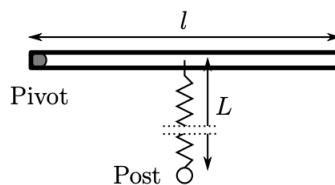
A mass on a frictionless table is attached to the midpoint of an originally unstretched spring fixed at the ends. If the mass is displaced a distance  $A$  parallel to the table surface but perpendicular to the spring, it exhibits oscillations. The period  $T$  of the oscillations

- (A) does not depend on  $A$ .
- (B) increases as  $A$  increases, approaching a fixed value.
- (C) decreases as  $A$  increases, approaching a fixed value.
- (D) is approximately constant for small values of  $A$ , then increases without bound.
- (E) is approximately constant for small values of  $A$ , then decreases without bound.

A uniform rod of length  $l$  lies on a frictionless horizontal surface. One end of the rod is attached to a pivot. An un-stretched spring of length  $L \gg l$  lies on the surface perpendicular to the rod; one end of the spring is attached to the movable end of the rod, and the other end is attached to a fixed post. When the rod is rotated slightly about the pivot, it oscillates at frequency  $f$ .

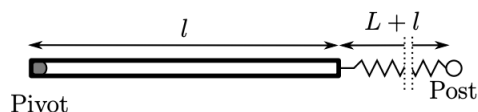


The spring attachment is moved to the midpoint of the rod, and the post is moved so the spring remains unstretched and perpendicular to the rod. The system is again set into small oscillations. What is the new frequency of oscillation?



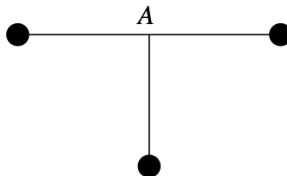
- (A)  $f/2$
- (B)  $f/\sqrt{2}$
- (C)  $f$
- (D)  $\sqrt{2}f$
- (E)  $2f$

The spring attachment is moved back to the end of the rod; the post is moved so that it is in line with the rod and the pivot and the spring is unstretched. The post is then moved away from the pivot by an additional amount  $l$ . What is the new frequency of oscillation?



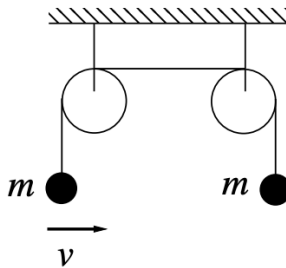
- (A)  $f/3$
- (B)  $f/\sqrt{3}$
- (C)  $f$
- (D)  $\sqrt{3}f$
- (E)  $3f$

Three identical masses are connected with identical rigid rods and pivoted at point  $A$ . If the lowest mass receives a small horizontal push to the left, it oscillates with period  $T_1$ . If it instead receives a small push into the page, it oscillates with period  $T_2$ . The ratio  $T_1/T_2$  is



- (A)  $1/2$
- (B)  $1$
- (C)  $\sqrt{3}$
- (D)  $2\sqrt{2}$
- (E)  $2\sqrt{5}$

Two particles of mass  $m$  are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity, and oscillates back and forth. The mass on the right

- (A) remains at rest
- (B) oscillates vertically, and with a net upward motion
- (C) oscillates vertically, and with a net downward motion
- (D) oscillates vertically, with no net motion
- (E) oscillates horizontally, with no net motion