

# Gravitation

January 29, 2024

## Gravitation Handout

### 0.1 Newton's Law of Universal Gravitation

When Newton first felt the apple fall down upon his head, he thought of an idea. An idea that there is a gravitational field created by the mass of an object. This field was called the gravitational field and led to the gravitational force equation which stated that the force  $F$  that a mass  $M$  imparts on a mass  $m$  a distance  $r$  away is equal to

$$F_g = \frac{-GMm}{r^2}$$

(the negative sign indicates that the objects are attracting each other) where  $G$  is a universal constant (approximately  $6.674 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$ ). The acceleration due to gravity  $g$  distance  $r$  away from a mass  $M$  is

$$g = \frac{-GM}{r^2}.$$

Oftentimes  $F=ma$  exam problems deal with the relationship of the gravitational force to various quantities such as the distance between both objects and the masses of the objects. From the force equation we can see that

$$F_g \propto r^{-2} Mm$$

where  $M$  and  $m$  are the masses of both objects.

### 0.2 Gravitational Potential Energy

#### 0.2.1 Derivation for AP C students

Potential energy is equal to the negative work done by a conservative force. As such using the gravitational force equation we can find an equation for the gravitational potential energy of a mass  $m$  based on its distance  $r$  from a mass  $M$ . Using  $F_g = \frac{-GMm}{r^2}$  and the fact that at an infinite distance away the potential energy is 0, we can use the fact that

$$U_g = - \int F_g \cdot dx$$

### 0.3 Kepler's First Law

to find the potential energy for a mass  $m$  at a distance  $r$  away.

$$U(r) = - \int_{\infty}^r \frac{-GMm}{r^2} \cdot dr$$

evaluating that integral we get that  $U(r) = \frac{-GMm}{r}$ . This is the potential energy of a mass  $m$  at a distance  $r$  away from a mass  $M$ .

$$U(r) = \frac{-GMm}{r}$$

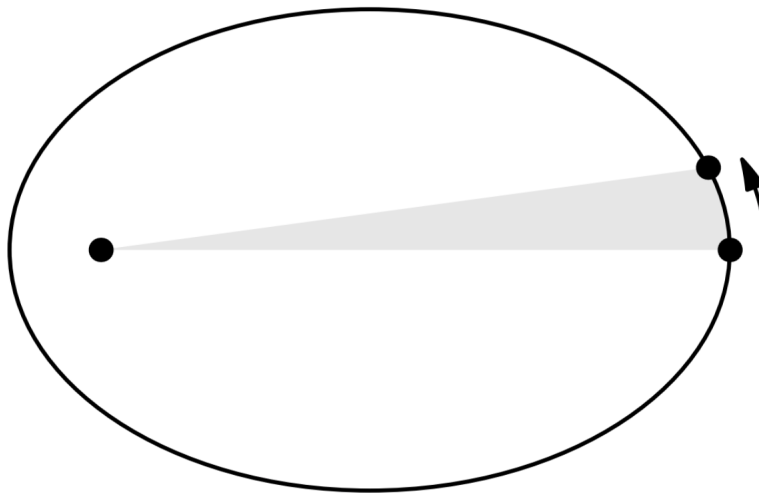
### 0.3 Kepler's First Law

Kepler's First Law states that the orbits of planets in our solar systems are elliptical in orbit and further goes on to state that the orbit of one object about another based on a gravitational field is always going to be either circular or elliptical in orbit. (Note: objects can have orbits that are not elliptical or circular such as comets.)

The generalization of Kepler's First Laws is that for a system of two objects the orbit of one object about the other is a conic section (parabola, hyperbola, circle, ellipse), with the center of mass at a focus. When one body is much more massive (such as the Sun and Earth), that body is approximately the focus. Kepler's Laws do not hold for more than 2 body systems...in fact scientists are still trying to figure out more than 2 body systems)

### 0.4 Kepler's Second Law

Kepler's Second Law states that a planet in gravitational orbit around a star traces out equal areas in equal times.



Once we cover angular momentum, we will be easily equipped to proving this statement but for now this proof is left as an exercise to the reader. The direct consequence of this

## 0.5 Kepler's 3rd Law

statement is that the dot product of the planet's velocity and distance from the star is conserved assuming that the planet experiences no other external forces that do not point in the direction of the position vector (a line that connects the planet to the star). So  $\vec{v}(t) \cdot \vec{r}(t) = k$  where  $k$  is a constant  $\vec{v}(t)$  is the velocity vector of the planet and  $\vec{r}(t)$  is the position vector which goes from the star to the planet at time  $t$ .

## 0.5 Kepler's 3rd Law

Kepler's 3rd Law states that if a planet is in an elliptical orbit around a star with mass  $M$  then the period of that elliptical orbit is

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}$$

where  $a$  is the semi major axis of the ellipse. This can also be written as

$$T^2 \propto a^3.$$

That means that no matter how wide the elliptical orbit is, the period remains the same.

## 0.6 Shell and Sphere Theory

The shell theorem states that if you have a mass  $m$  that has the shape of a shell with radius  $r$ , the gravitational acceleration at any point inside the shell is 0. However for a point a distance  $R$  away from the center of the shell where  $R > r$ , the gravitational acceleration at that point is

$$a_g = \frac{GM}{R^2}$$

. This means that if you wanted to find the gravitational acceleration of a point outside of the shell you can just consider the shell to be a point mass located at the center of the shell.

The sphere theorem states that if you have a mass  $m$  that has the shape of a sphere with radius  $r$  and you wanted to find the gravitational acceleration of a point a distance  $R$  away where  $R > r$  then

$$a_g = \frac{GM}{r^2}$$

. This means that you can consider the sphere to be a point mass located at the center of the sphere while you are calculating the gravitational acceleration.

The proof for shell theorem is quite intuitive and we encourage you to search it up.

## 0.7 Virial Theorem

Virial Theorem states that if the orbit of one of the objects is elliptical or circular then the kinetic energy of the object at a specific point in time is equal to the negative of half the Potential Energy at that point in time.

$$K = -\frac{1}{2}U$$

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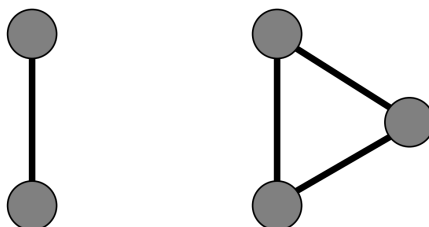
# 1 Gravitational Problems

## 1.1 Note

Gravitation is an interesting subject with a lot of concepts and techniques associated with its problems. This handout serves as a primer for gravitation, and we highly suggest taking Morin or Reznick and Halliday and going through the chapter while doing the problems on Gravitation.

## 1.2 Gravitational Problems

12. You are given a large collection of identical heavy balls and lightweight rods. When two balls are placed at the ends of one rod and interact through their mutual gravitational attraction (as is shown on the left), the compressive force in the rod is  $F$ . Next, three balls and three rods are placed at the vertexes and edges of an equilateral triangle (as is shown on the right). What is the compressive force in each rod in the latter case?



- (A)  $\frac{1}{\sqrt{3}}F$
- (B)  $\frac{\sqrt{3}}{2}F$
- (C)  $F$
- (D)  $\sqrt{3}F$
- (E)  $2F$

Say that an object is in a circular orbit of radius  $R$  around the Earth which has a mass  $M_E$ . What is the speed of the object in terms of  $R$  and  $M_E$ ?

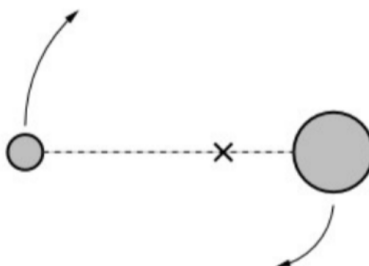
A 1000 Kg satellite is in synchronous orbit around planet earth. The period of this synchronous orbit matches the rotation of the earth around its axis, assumed to be 24 hours, so that the satellite appears stationary. What is the orbital radius of the satellite?

17. A spherical cloud of dust in space has a uniform density  $\rho_0$  and a radius  $R_0$ . The gravitational acceleration of free fall at the surface of the cloud due to the mass of the cloud is  $g_0$ .

A process occurs (heat expansion) that causes the cloud to suddenly grow to a radius  $2R_0$ , while maintaining a uniform (but not constant) density. The gravitational acceleration of free fall at a point  $R_0$  away from the center of the cloud due to the mass of the cloud is now

- (A)  $g_0/32$
- (B)  $g_0/16$
- (C)  $g_0/8$
- (D)  $g_0/4$
- (E)  $g_0/2$

Two stars orbit their common center of mass as shown in the diagram below. The masses of the two stars are  $3M$  and  $M$ . The distance between the stars is  $d$ .



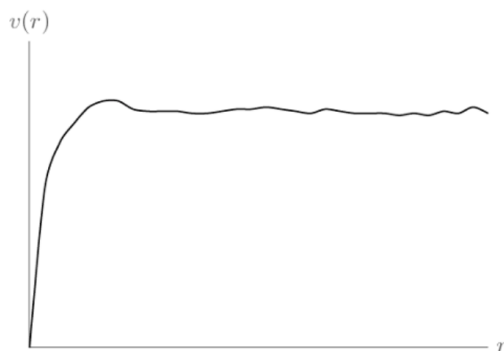
Determine the period of orbit for the star of mass  $3M$ .

- (A)  $\pi\sqrt{\frac{d^3}{GM}}$
- (B)  $\frac{3\pi}{4}\sqrt{\frac{d^3}{GM}}$
- (C)  $\pi\sqrt{\frac{d^3}{3GM}}$
- (D)  $2\pi\sqrt{\frac{d^3}{GM}}$
- (E)  $\frac{\pi}{4}\sqrt{\frac{d^3}{GM}}$

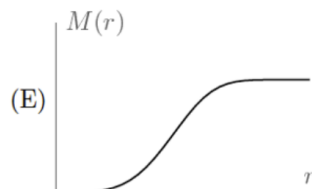
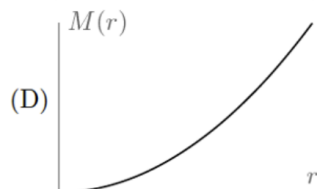
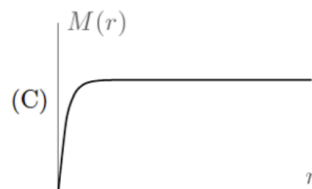
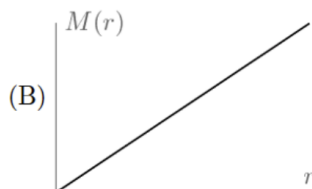
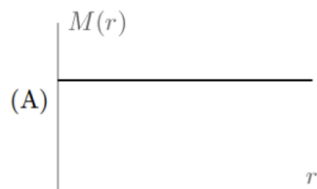
A uniform circular ring of radius  $R$  is fixed in place. A particle is placed on the axis of the ring at a distance much greater than  $R$  and allowed to fall towards the ring under the influence of the ring's gravity. The particle achieves a maximum speed  $v$ . The ring is replaced with one of the same (linear) mass density but radius  $2R$ , and the experiment is repeated. What is the new maximum speed of the particle?

- (a)  $\frac{1}{2}v$
- (b)  $\frac{1}{\sqrt{2}}v$
- (c)  $v$
- (d)  $\sqrt{2}v$
- (e)  $2v$

15. The following plot shows the speed  $v(r)$  at which stars orbit about the center of a galaxy, as a function of their distance  $r$  from the center.



Assuming the galaxy has a spherically symmetric mass distribution, which plot best shows the mass of the galaxy  $M(r)$  enclosed within radius  $r$ ?



## 2 Additional Derivations

### 2.0.1 What is the change in potential energy for small changes in $r$ ?

Say that we have a mass  $m$  a distance  $r$  away from a mass  $M$ . As we proved before its potential energy would be  $U(r) = \frac{-GMm}{r}$ . Now let's say that we moved the object a distance  $\Delta r$  away where  $\frac{\Delta r}{r} \rightarrow 0$ . Then we can say that the mass  $m$  is now a distance of  $r + \Delta r$  away from mass  $M$ . What is its potential energy now? Well...it would just be  $U(r) = \frac{-GMm}{r + \Delta r}$ . But then what would be the change in its potential energy after the movement. The change in potential energy  $\Delta U = \frac{-GMm}{r + \Delta r} - \frac{-GMm}{r} = \frac{GMm}{r} - \frac{GMm}{r + \Delta r}$  simplifying the expression by multiplying by  $(r + \Delta r)r$  we get  $\Delta U = \frac{GMm\Delta r}{r(r + \Delta r)} = \frac{GMm}{r} \frac{\Delta r}{1 + \frac{\Delta r}{r}}$ . Using Taylor Series Approximations, we find that  $\Delta U = \frac{GMm}{r} \frac{\Delta r}{1 + \frac{\Delta r}{r}} = \frac{GMm}{r^2} \Delta r$ . We know that the magnitude of the acceleration due to gravity for a mass  $M$  a distance  $r$  away is  $\frac{GM}{r^2}$ . If we say that the acceleration due to gravity is equal to  $g$  then  $\Delta U = \frac{GMm}{r^2} \Delta r = mg\Delta r$  for  $\Delta r$  such that  $\frac{\Delta r}{r} \rightarrow 0$  ( $\Delta r$  comparatively smaller than  $r$ ). This is why we approximate changes in potential energy with  $mgh$  for problems dealing with small changes in height.